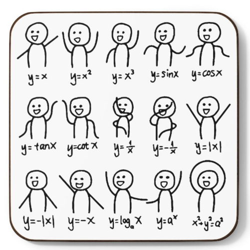
Grade 9/10 Math Circles
February 22
The Shape of You

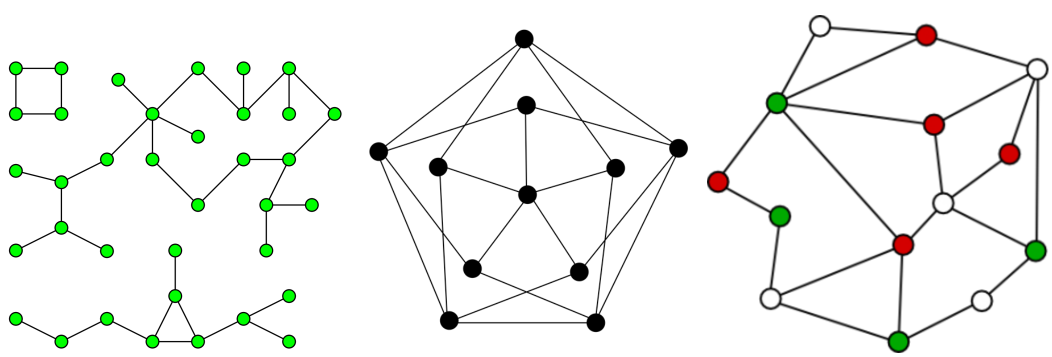
Okay folks, I'm afraid this is our last ride together... But there are lots more shapes to come! And feel free to reach out to me anytime at **mgusak@uwaterloo.ca** for more cool stuff.

Graphs!

Let's study graphs. Not the kinds of graphs you learn in high school...



Nope, we will define a **graph** to be a bunch of dots with (possibly) some lines between them. For example:

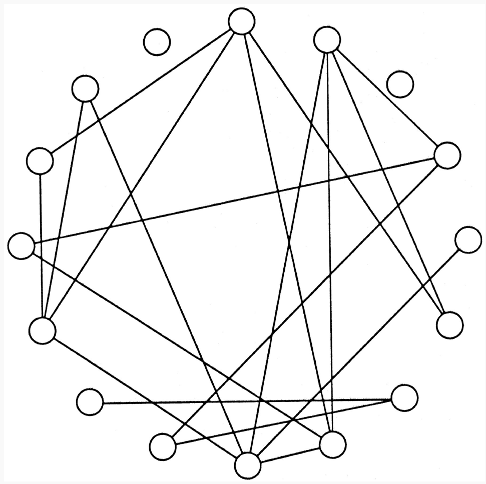


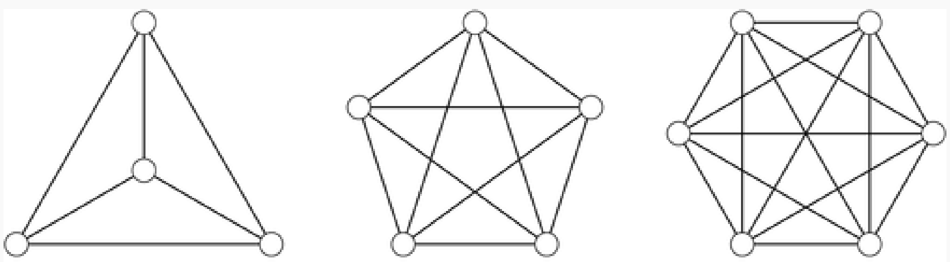
are all different examples of graphs. The lengths of the lines and the exact locations of the dots don't matter[1]. So we've been (accidentally) drawing graphs for the last two weeks!

[1] Unless you call yourself a geometric graph theorist. But then I've got other questions for you.

Colouring Challenge

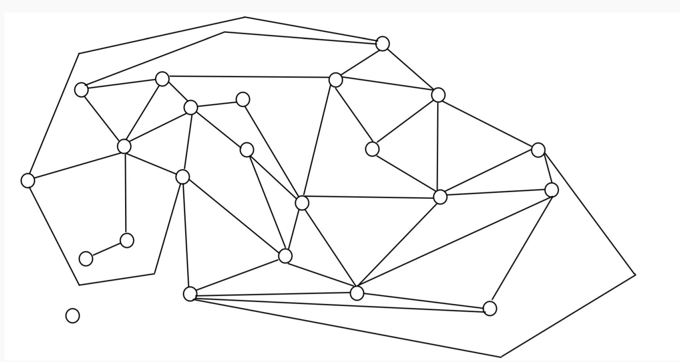
Find a way to "color" each of the dots in the graphs below with *as few colors as possible* so that if two dots are connected with a line then they CANNOT be the SAME color.





K_4 K_5 K_6

For this example (and forever onwards) I will also allow "curved" lines (or lines that bend) to connect different dots.



How can you show that your solution is "the best" (ie. that it uses the smallest number of colors possible?)

Computers in Math

If you do a bunch of graph coloring examples, you might notice that it's a little easier to color a graph using less colors if the lines in the graph generally cross less. This is an important idea!

A graph is called **planar** if we can redraw it (keeping the number of dots and how they are connected with lines *the same*), BUT without any line crossings. Try to label all the graph examples I've given you above as **planar** or **non-planar** (note: a planar graph might *look* non-planar if you accidentally drew it with line crossings).

FOUR COLOUR THEOREM: Every *planar* graph can be coloured using *at most* 4 colors.

Note: This is a huge theorem!! I *highly* suggest you try proving it yourself and see how far you get. It was first (correctly) proven in 1976 and was *super controversial*. That's because Appel and Haken, the authors of the proof, had to use a computer to verify a large number of cases that is infeasible to be checked by humans. Computers had only been around for a little bit of time, and mathematicians were skeptical about accepting a computer-assisted proof that they couldn't actually check.

What do you think? Are computer-assisted proof just as valid as human-written proofs, or can you argue that leaving a proof to just "trusting a computer" removes some of the beauty of mathematics?

Let's try to prove an easier version of this theorem! The SIX COLOR THEOREM. As you can guess, this version of the theorem says that every graph can be colored using at most *six* colors, so that no two dots that are joined by a line are the same color. The proof goes something like this:

SIX COLOR THEOREM Proof:

Definition: I'm going to call the number of lines touching a dot the **degree** of that dot.

Handshake Lemma: In any graph, the sum of the degrees of all the dots is equal to twice the number of lines. Try to prove this yourself!

Planar Fact: Take an arbitrary planar graph. Let L be the number of lines and D be the number of dots. The $L \leq 3D - 6$. This inequality tells us that planar graphs can't have "too many lines", and for brevity I'll blackbox this fact.

Planar Lemma: Every planar graph contains a dot who's degree is less than or equal to five. *Proof:* Take an arbitrary planar graph, and suppose *for a contradiction* that every one of its dots has degree at least six. So the sum of the degrees of all the dots is at least six times the number of dots. By the Handshake Lemma, the sum of the degrees of all the dots is equal to twice the number of lines. Using the above notation,

$$2L \geq 6D$$

Therefore,

$$L \geq 3D \geq 3D - 6$$

This contradicts our fact! So we must've had a dot with a degree less than six.

Proof of Six Color Theorem: Take an arbitrary planar graph. From the Planar Lemma, we know there is a dot of degree five or less. That means, if we take that dot out (and all the lines connected to it) and if we could somehow color the rest of the graph with less than six colors, then we could definitely put that dot back in and just color it a different color than each of its (at most) five neighbours. Then we'd be done! So the problem is now to color that smaller graph...

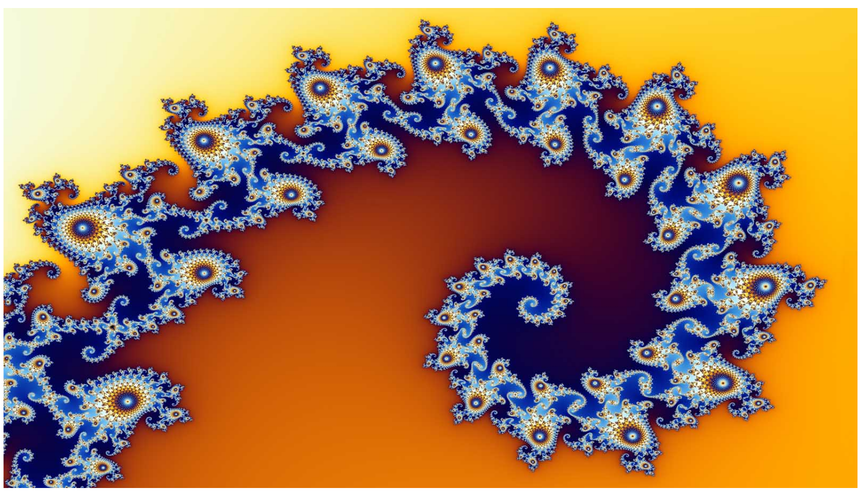
By taking out that dot, we are left with a *smaller graph that is still planar.* So repeat the argument! Keep taking out dots. There are only finitely many dots to take out, so eventually we'll have just one dot left, and that's obviously six-colorable. Then add the dots and lines you just removed in reverse order, colouring as you go, until you get back the original graph!

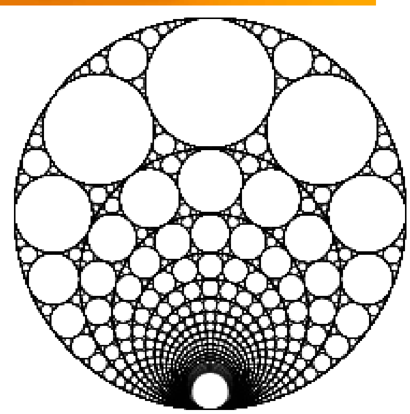
To Infinity and Beyond!

Here are some more funny shapes. Can you think of a shape with a *finite* perimeter and *infinite* area? Can you think of a shape with a *finite* area and *infinite* perimeter? Are these even possible??

Possible contenders to answer these questions are shapes called “fractals”. **Fractals** are shapes that “look like themselves” wherever you zoom in. They are usually built in steps, because constructing a fractal takes “infinitely many steps”.

Here are some examples of fractals:



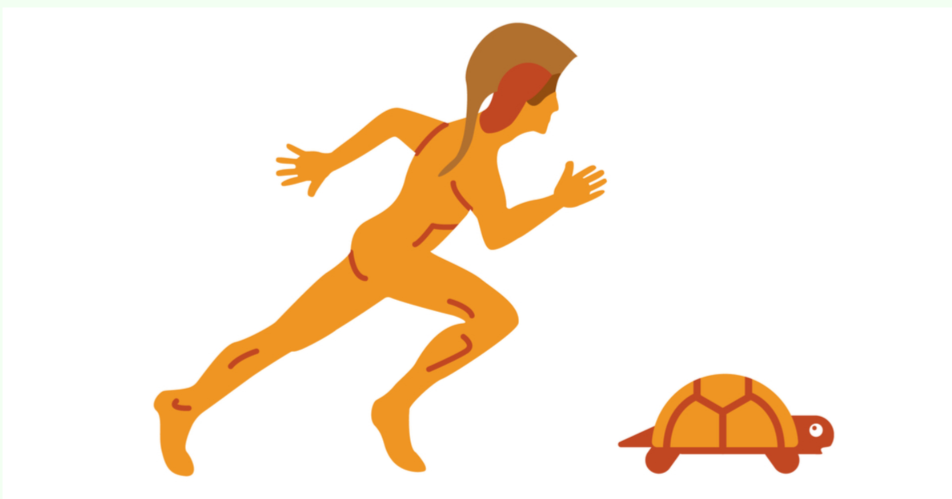


To address infinity, we better start with the oldest paradox in the books:

Zeno's Paradox

The Greek mythology Achilles was known to be fast, while tortoises are known to be.. well, slow. A tortoise who had half the speed of Achilles once dared him to a race, and claimed that it could beat Achilles if he gave the tortoise a headstart of 100 metres. Here's the tortoise's argument:

"Let me start 100 metres ahead of you. Then, by the time you get to the 100 metre mark, I will have gone on a further 50 metres, and will still be ahead. By the time you run the next 50 metres, I will have gone on a further 25, and will still be ahead. By the time you run the next 25, I will have gone on further, and will still be ahead. Extrapolating this to infinity, this means that I will always be ahead of you and will win the race".



Sounds like the tortoise has got a point. But suppose Achilles runs at 10 metres per second and the tortoise moves at 5 metres per second. In 100 seconds, Achilles will have travelled $(100s)(10m/s) = 1000$ metres, but the tortoise will have only travelled $100m + (100s)(5m/s) = 600$ metres, meaning Achilles will already have been way ahead of the tortoise and could obviously beat it in a race!

So is this a paradox in math? physics? logic? reality?

How do you explain this?

Adding Infinitely Many Things Together

It's hard to add infinitely many things together. But mathematicians came up with *one way* to make sense of "adding infinitely many things together".

Adding Adding Adding Adding Addi...

Suppose we want to figure out what this sum equals:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = ?$$

No computer or calculator or human can do infinitely many additions, so if we stop at "finitely many" terms we'll only get an approximation of what the sum equals. Let's take a look at these approximations:

$1 = 1$
$1 + \frac{1}{2} = \frac{3}{2}$
$1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

The pattern $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$ looks like it's heading to the number 2, so let's just define the sum to be that!

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

More generally, we can derive the formula

$$1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$$

Does this work for all r? It might be ridiculous to say that

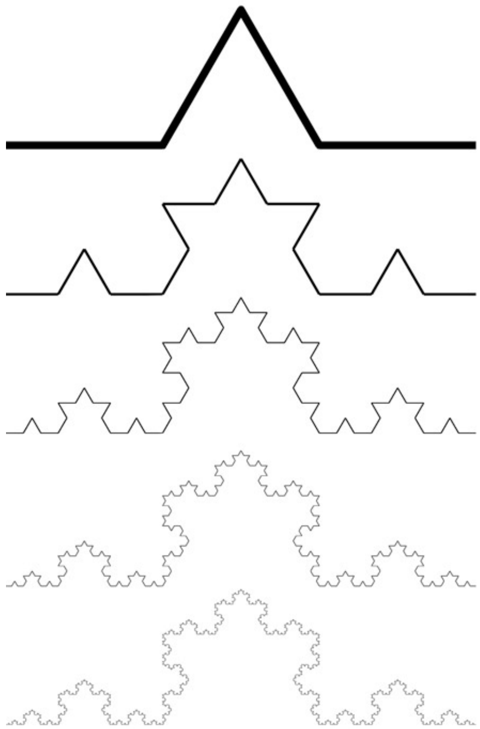
$$1 + 2 + 4 + 8 + \ldots = \frac{1}{1-2}$$

We say that this formula only works when $|r| < 1$. (**Note**: why should this make sense?)

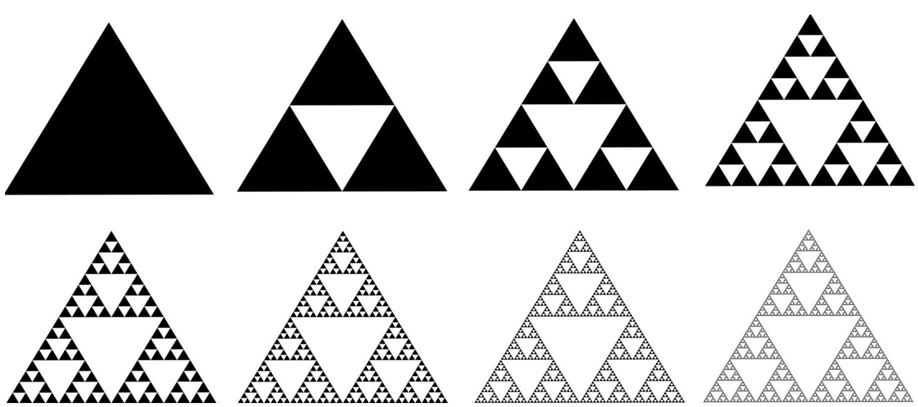
This formula should be enough to compute some cool fractal math!

Can you figure out the areas and perimeters of the shapes on the following pages? The starting area/length can be defined to be 1 to make things easier:

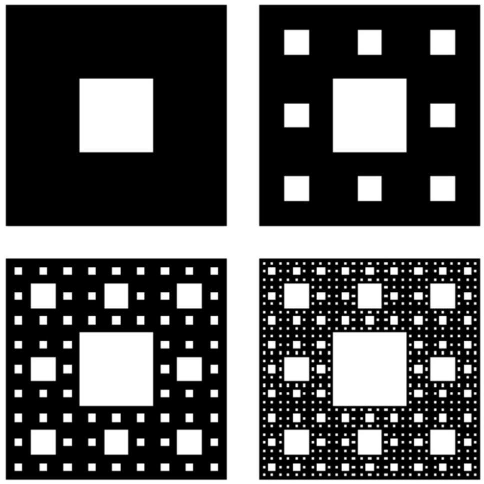
Koch snowflake



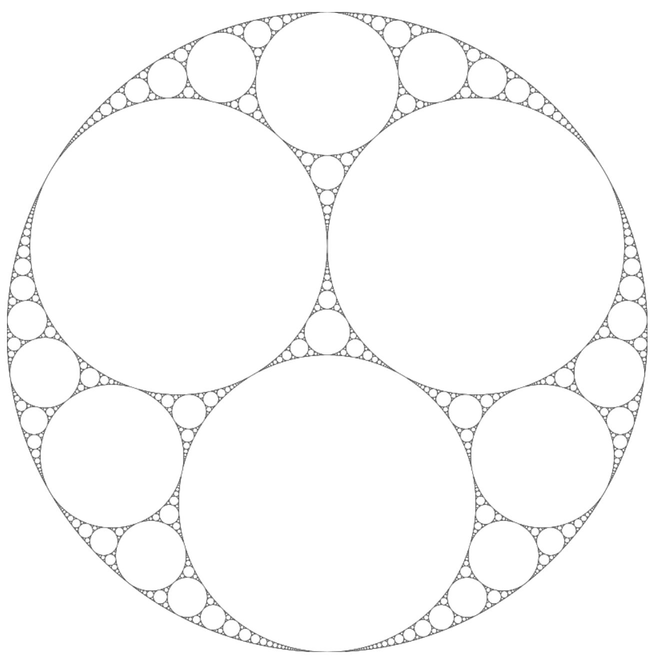
Sierpinski gasket



Sierpinski carpet



Apollonian gasket



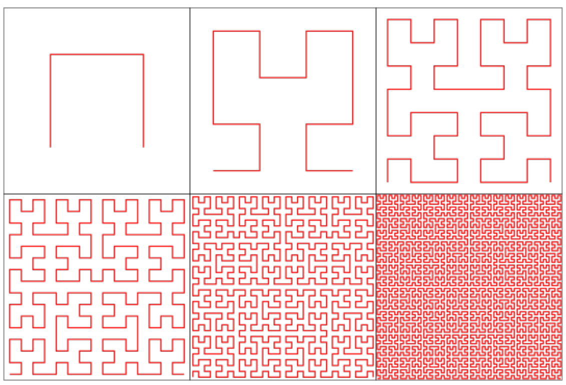
DIY Fractals

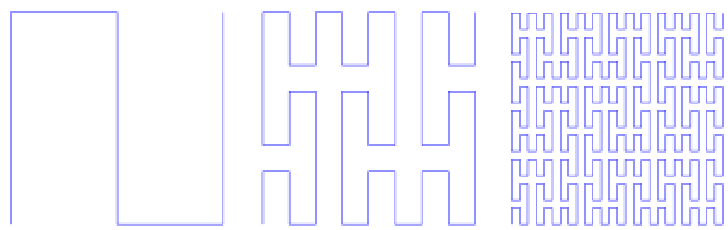
In the space below, try constructing your own fractals! What patterns can you make? Are the perimeters/areas finite or infinite? Can you generalize any of the previous fractals to three dimensions?

Hilbert's Hat

The last topic I'd like to mention are the infamous **space filling curves**. A **space filling curve** is a continuous curve (you could draw it with a pencil) that never crosses itself or lies on top of itself but still crosses every single point in 2D (or 3D) space.

Here are two famous examples:





DIY Space-Filling Curve
Same challenge! Make your own space-filling curves, and then try to extend it (or the ones above) to three dimensions...